mquantreg: An R package for estimating generalized linear M-quantile regression models

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General information about the mquantreg package

- Package **mquantreg** is still under development
- Regression package for the estimation of linear and generalized linear M-quantile regression models
  - Continuous data
  - Binary data
  - Count data (poisson, quasi-poisson, negative binomial)
- Hypothesis testing via Wald- and Likelihood-Ratio Tests
- Estimation of q-scores (variable selection; further use in SAE-methods)
- Model diagnostics using diagnostic plots

Short excursion into theory (1)

What is M-estimation?

- Huber (1964) formalised a approach to robust estimation of location parameters called M-estimation:

$$\hat{\theta} = \arg\min_{\theta} \left(n^{-1} \sum_{i=1}^{n} \rho(x_i; \theta) \right)$$

or if $\rho(\cdot)$ is differentiable and convex a solution is easier found with **influence function** $\psi(x; \theta)$:

$$n^{-1} \sum_{i=1}^{n} \psi(x_i; \hat{\theta}) = 0, \text{where } \psi(x; \theta) = \frac{\partial}{\partial \theta} \rho(x; \theta)$$

- examples: $\rho(x; \bar{x}) = (x - \bar{x})^2$, $\rho(x; \widetilde{x}) = |x - \widetilde{x}|$

Short excursion into theory (2)

Huber Proposal 2 for $\psi(x;\theta)$:

$$\psi_k(x-\theta) = \begin{cases} x-\theta, & \text{if } |x-\theta| < k \\ k\,\text{sgn}(x-\theta), & \text{if } |x-\theta| \geq k \end{cases}$$

or alternatively $\psi_k(x-\theta) = \max(-k, \min(x-\theta, k))$ with **tuning constant** k, which has to be chosen beforehand.

- k determines the robustness of the estimator
- Huber estimator is equivalent to the mean for $k \to \infty$ and to the median for $k \to 0$

Short excursion into theory (3)

- Generalization of the concept to the regression case (Huber, 1973)
- Further development by Breckling and Chambers (1988) to M-quantile regression for continuous data:

Influence Function (depending on q and k):

$$\psi_{q,k}(y - MQ_{q,k}) = 2[(1-q)I_{y \leq MQ_{q,k}} + qI_{y > MQ_{q,k}}]\psi_k(y - MQ_{q,k})$$

Estimating equations (depending on q and k):

$$n^{-1}\sum_{i=1}^{n}\psi_{q,k}(y_i - \hat{MQ}_{q,k}(x_i))x_i = 0, \text{ where } \hat{MQ}_{q,k}(x_i) = x_i^T\hat{\beta}_{q,k}$$

Short excursion into theory (4)

- M-quantiles are not scale-equivariant: scale parameter $\sigma_{q,k}$ has to estimated at the same time as regression coefficients
- The parameters can be estimated via IRLS or ML using Asymmetric Least Informative (ALI) distribution (Bianchi et al. 2018)
- Categorical data
  - Generalizations of estimation approach to robust generalised linear models by Cantoni and Ronchetti (2001)
  - Binary: Chambers, Salvati and Tzavidis (2016)
  - Poisson: Tzavidis, Ranalli et al. (2015)
  - Negative Binomial: Chambers, Dreassi and Salvati (2014)

Short excursion into theory (5)

Q-scores:

- Application of M-quantiles by Kokic et al. (1997)
- Can be found by solving for q_i^*, which solves

$$MQ_{q_i^*,k}(x_i) = y_i$$

- Observations with high q-scores have values high values y_i conditional on x_i
- Q-scores can be used in a lot of different ways in a M-quantile regression context
  - Model diagnostics
  - Variable selection (Dawber, 2017)
  - Useful for example in Small Area Estimation for the estimation of pseudo-random effects (Chambers and Tzavidis (2006))

Estimation

Example for continuous data using a sample of size 1000 from the German EUSILC PUMD (Public Use Microdata) from 2013:

```
fit <- mquantreg(formula = income ~ income_self_empl +
    income_empl + old_age_benefits + sex,
    q = c(0.1, 0.5, 0.7),
    k = 1.345,
    data = dat,
    method = "continuous",
    scale.estimator = "Mad",
    cont = "lik",
    compute.qscores = FALSE)
```

Print and Summary functions (1)

```
> print(fit)
Call:
mquantreg(formula = income ~ income_self_empl + income_
    empl +old_age_benefits + sex, data = dat, q = c
    (0.1, 0.5, 0.7), method = "continuous", scale.
    estimator = "Mad", compute.qscores = FALSE, cont =
    "lik")

Coefficients:
     q   (Intercept) income_self_empl income_empl
[1,] 0.1    4.936694     0.0004245352   0.4365187
[2,] 0.5   12.418848     0.0005556530   0.5109797
[3,] 0.7   15.776415     0.0006699692   0.5382013

old_age_benefits sexFemale
0.4866095        1.523643
0.4279946        2.754370
0.4199769        3.209267
Degrees of freedom: 1000 total; 995 residual
```

Print and Summary functions (2)

```
> summary(fit)
Estimation for m-quantile  q = 0.5 with k =  1.345
n = 1000

Coefficients:
                coefficients   Std. Error
(Intercept)       12.418847514 5.827978e-01
income_self_empl   0.000555653 3.709961e-05
income_empl        0.510979679 1.179632e-02
old_age_benefits   0.427994581 3.231350e-02
sexFemale          2.754369615 6.575171e-01
Estimator for scale parameter sigma:  9.593204

Residuals:
-27.30647 -6.399448 -0.7536041 1.611043 6.858612
    102.6775

Proportion of residuals smaller than 0: 0.539
Residuals bounded at +- 12.90286
Proportion of Huberised residuals: 17.7 %
Pseudo R-squared (Bianchi et al, 2015):  0.1356437
```

Variable Selection

```
fit_sel <- stepmq(fit)
Starting model:
income ~ income_self_empl + income_empl + old_age_
    benefits + sex

                        Df      Corr.
<none>                  NA 1.0000000
- sex                    1 0.9862613
- income_self_empl       1 0.9757906
- old_age_benefits       1 0.9072897
- income_empl            1 0.7163004
                        Df      Corr.
<none>                  NA 1.0000000
- income_self_empl       1 0.9761506
- old_age_benefits       1 0.9162316
- income_empl            1 0.7169594
                        Df      Corr.
<none>                  NA 1.0000000
- old_age_benefits       1 0.9241252
- income_empl            1 0.7323867
```

Hypothesis Testing (1)

Wald tests for single coefficients:

```
> mqwald(fit, id.beta = c(3,4), each = TRUE, q = 0.5)
mqwald(fit, id.beta = c(3,4), each = TRUE, q = 0.5)
Call:
mqwald(object = fit, id.beta = c(3, 4), q = 0.5, each =
    TRUE)

Wald tests for single coefficients at q-value(s) 0.5
H0:[1] 0 0

test-statistic:
      income_empl old_age_benefits
[1,]    1876.353          175.4319

degrees of freedom
[1] 1 1

p-values
       income_empl old_age_benefits
[1,]             0                0
```

Hypothesis Testing (2)

Likelihood-Ratio Omnibus-Test:

```
> mqLRT(fit)
Call:
mqLRT(object = fit)

Estimated coefficients to be restricted to zero in
    reduced model:

    [...]

Result of Likelihood-Ratio test(s):
        q-value(s)  Chi^2  df   p-value(s)
[1,]           0.1  686.3380  4           0
[2,]           0.5  986.5277  4           0
[3,]           0.7  812.1397  4           0

The likelihood-ratio test gives significant results
    at a level alpha of 0.05 for the m-quantile(s)
    0.1, 0.5, 0.7 .
```

Graphical Diagnostics (1)

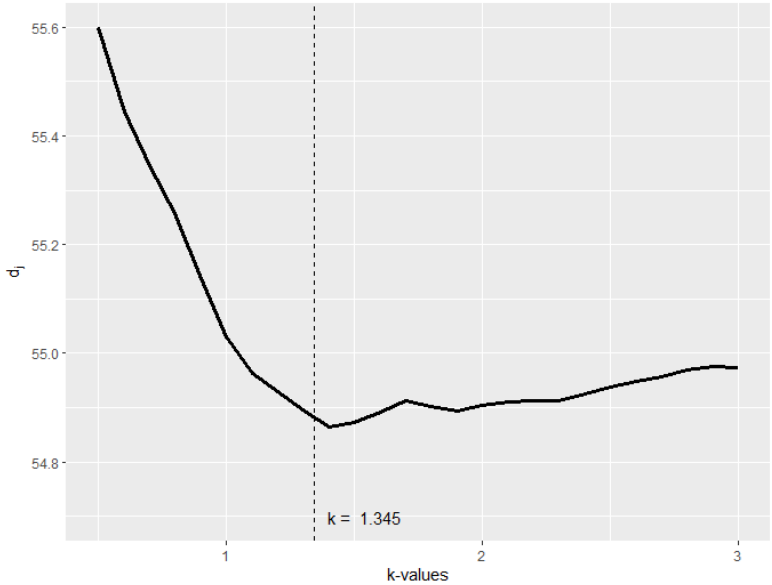
The packages includes the following ploting options (based on Dawber (2017)):

- "fit.res" - Fitted vs. Residuals Plot
- "prop.res" - Proportion of Residuals Smaller than 0 vs. Fitted Values
- "fitted.observed" - Fitted vs. Observerd Values
- "optk" - Comparing the Residual Distribution with Normal Distribution
- "coef.q" - Coefficients over q
- "prop.huber.k" - Proportion of Huberised residuals over k
- "prop.huber.q" - Proportion of Huberised residuals over q

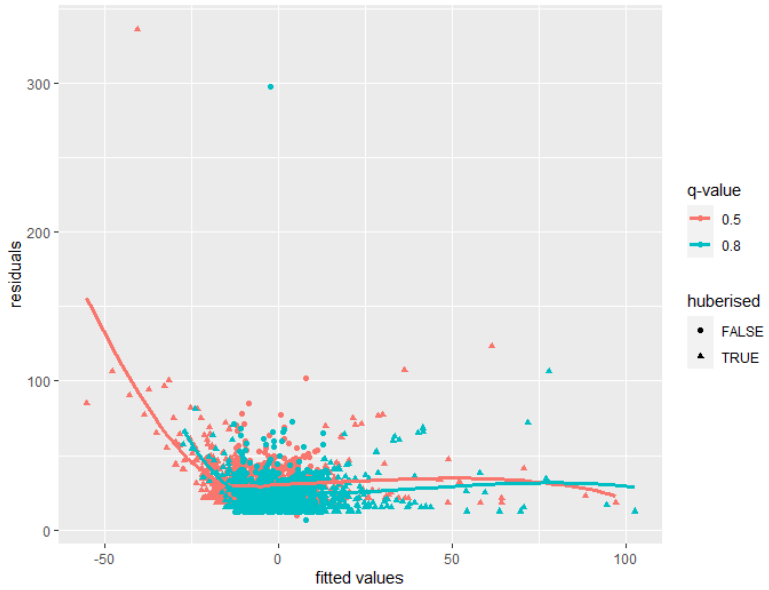
Graphical Diagnostics (2)

```
1    > plot(fit, plottype = "optk")
2    > plot(fit, plottype = "fit.res", add.qvals = c(0.8))
```

Optimal k



Fitted vs. Residuals



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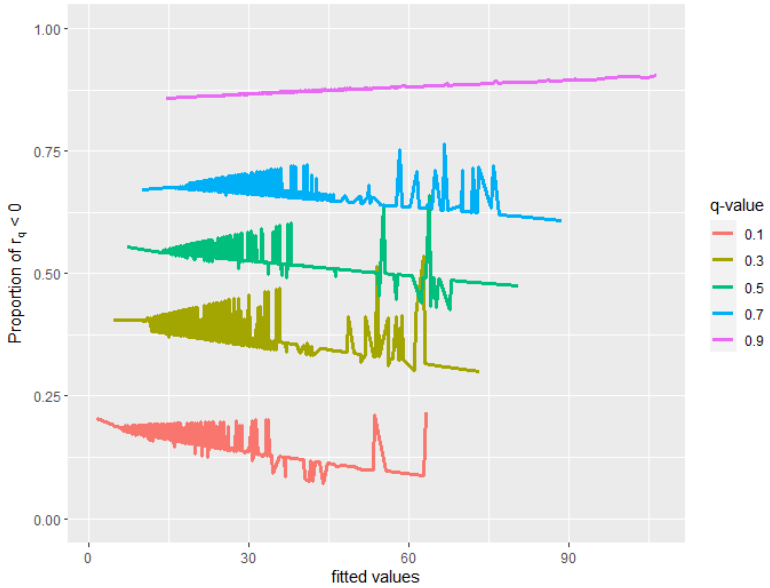
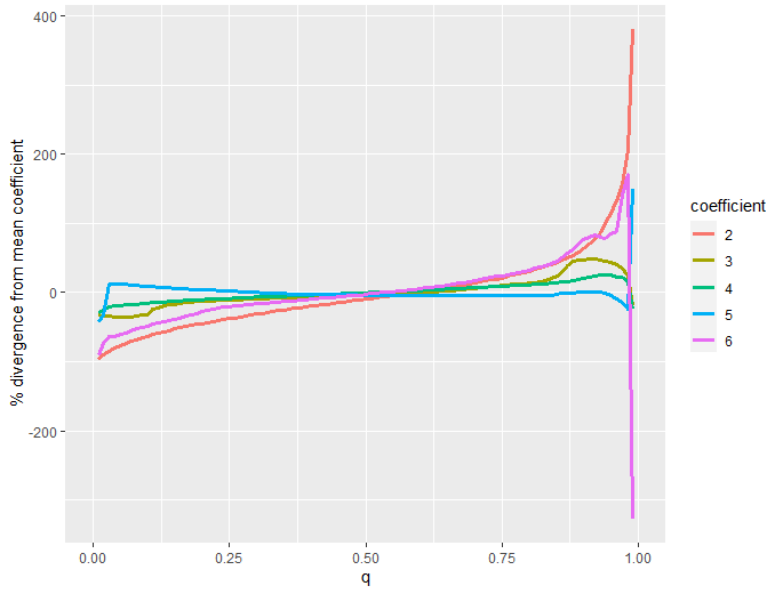
Graphical Diagnostics (3)

```
1    > plot(fit, plottype = "prop.res", add.qvals = c(0.1,0.3,0.7,
         0.9))
2    > plot(fit, plottype = "coef.q")
```

Fitted vs. Proportion negative Residuals



Coefficients over q-values



Using simulated data under contamination:

```
1       fit_binom <- mquantreg(formula = Y ~ x1 + x2,data =
            dat, q  = 0.5,
2           method = "binom")
```

Summary for simulated example

```
Call:
mquantreg(formula = "Y ~ x1 + x2", data = df, q =
    0.5, method = "binom")
Estimation for m-quantile  q = 0.5 with k =  1.6
n = 1000
Coefficients:
            coefficients Std. Error
(Intercept)   0.09966344 0.06557439
x1            0.36810515 0.06782330
x2            0.38303418 0.06855655
Estimator for scale parameter sigma:  0.4971013

Residuals:
-0.7280652 -0.4848358 0.2357902 0.001179976 0.4516844
     0.795725

Proportion of residuals smaller than 0: 0.475
Residuals bounded at +- 0.6686012
Proportion of Huberised residuals: 7.9 %
---
Estimation with Robust Quasi-Likelihood Estimation
```

Conclusion

- M-quantile regression is a field of continued research
- These methods are helpful not just as regression models, but can be applied in other contexts as for example SAE
- Package **mquantreg** can be the foundational package for further use of M-quantile regression
- In its current state package **mquantreg** already has the functionality users are accustomed to from other regression packages (i.e. summary(), print(), plot() based on classes)

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Thank you for your attention

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